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UNIVERSITY OF WATERLOO

Examination
Final
Spring 2024
SYDE 311

Times: Tuesday 2024-08-06 at 12:30 to 15:00 (3PM)
Duration: 2 hours 30 minutes (150 minutes)
Exam ID: 5734419
Sections: SYDE 311 LEC 001
Instructors: Stephen New

Name (print): ______________________

Signature: ______________________

ID Number: ______________________

Instructions:

1. Place your name, signature and ID number, on the lines provided above.

2. This test contains 8 pages, including this cover page and two pages at the end for extra space, if needed.

3. No calculators or any other electronic devices are allowed.

4. Answer all 5 questions; all questions will be given equal value.

5. Provide full explanations with all your solutions. If you run out of space then continue on the last pages.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

[10] **1:** (a) For the pair of ODEs $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2 + y - 1 \\ 2y - xy \end{pmatrix}$, find all the equilibrium points.

(b) For $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy + x \\ y^2 - x \end{pmatrix}$, determine whether $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is attracting or repelling.

(c) For $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy^2 \\ yx - y \end{pmatrix}$, find a conserved quantity $H(x, y)$.

[10] **2:** (a) Find the 4^{th} Taylor polynomial at 0 for the solution to the IVP $y' + xy = 1 + x^2$.

(b) Consider the ODE $x^2y'' + x^2y' + (x-2)\,y = 0$. Following Frobenius' method, find two values of r such that the DE has a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, then find an exact, closed form formula for a solution with $r = -1$ and $c_0 = 1$.

[10] **3:** (a) Find the Fourier series for the 2π-periodic function f with $f(x)=|x|$ for $-\pi\leq x\leq\pi$.

(b) Let f be the 2π-periodic function with $f(x) = -1$ for $-\pi \leq x < 0$ and $f(x) = 1$ for $0 \leq x < \pi$. The Fourier series of f is given by $\sum\limits_{n \text{ odd}} \frac{4}{\pi n} \sin nx$. By evaluating at $x = \frac{\pi}{2}$, and by using Parceval's identity, find $R=\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$, $S=\sum\limits_{k=0}^{\infty} \frac{1}{(2k+1)^2}$, and $T=\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

[10] **4:** (a) Find the solution $u(x,y)$ to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 \leq x \leq 2$ and $0 \leq y \leq 1$ which satisfies the boundary conditions $u(x,0) = x$, $u(x,1) = 1$, $u(0,y) = y$ and $u(2,y) = 2 - y$.

(b) Find the solution $u = u(x,t)$ to the heat equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ for $0 \leq x \leq 2$ and $t \geq 0$, satisfying the insulated ends condition $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(2,t) = 0$ for all $t \geq 0$ and the initial condition $u(x,0) = 2\cos^2(\pi x)$ for all $0 \leq x \leq 2$.

(c) Find all negative values $k < 0$ for which there exists a non-zero solution to the ODE $y'' = ky$ for $y = y(x)$ with $y(0) = 0$ and $y'(2) = 0$.

[10] **5:** (a) Use Euler's method with step size $h = \Delta t = \frac{1}{2}$ to approximate the point $(x(1), y(1))$ when $(x(t), y(t))$ is the solution to $x' = y^2 - 1$ and $y' = x + y$ with $x(0) = y(0) = 0$.

(b) Find the first approximation $\binom{x_1}{y_1}$ when Newton's method is used to approximate a solution to the equation $\binom{2+x-y^2}{2+y-x^2} = \binom{0}{0}$ starting with $\binom{x_0}{y_0} = \binom{1}{-1}$.

(c) Find the weights w_0, w_1 and w_2 for the Newton-Cotes quadrature rule using the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 3$ in the interval $[0, 3]$ to give $\int_0^3 f(x)\,dx \cong \sum_{k=0}^{2} w_k f(x_k)$.

Use this page to continue solutions if you require additional space.
If you do, then clearly indicate which questions you are continuing.

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